Geometric Algebra

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Questions

- How are dot and cross products related?
- Why do cross products only exist in 3D?
- Generalize "cross products" to any dimension?
- Is it possible to divide by a vector?
- What does an imaginary number look like?
- Complex have two, but Quaternions have four. Why?
- Why do quaternions rotate vectors?
- Generalize quaternions to any dimension?

History

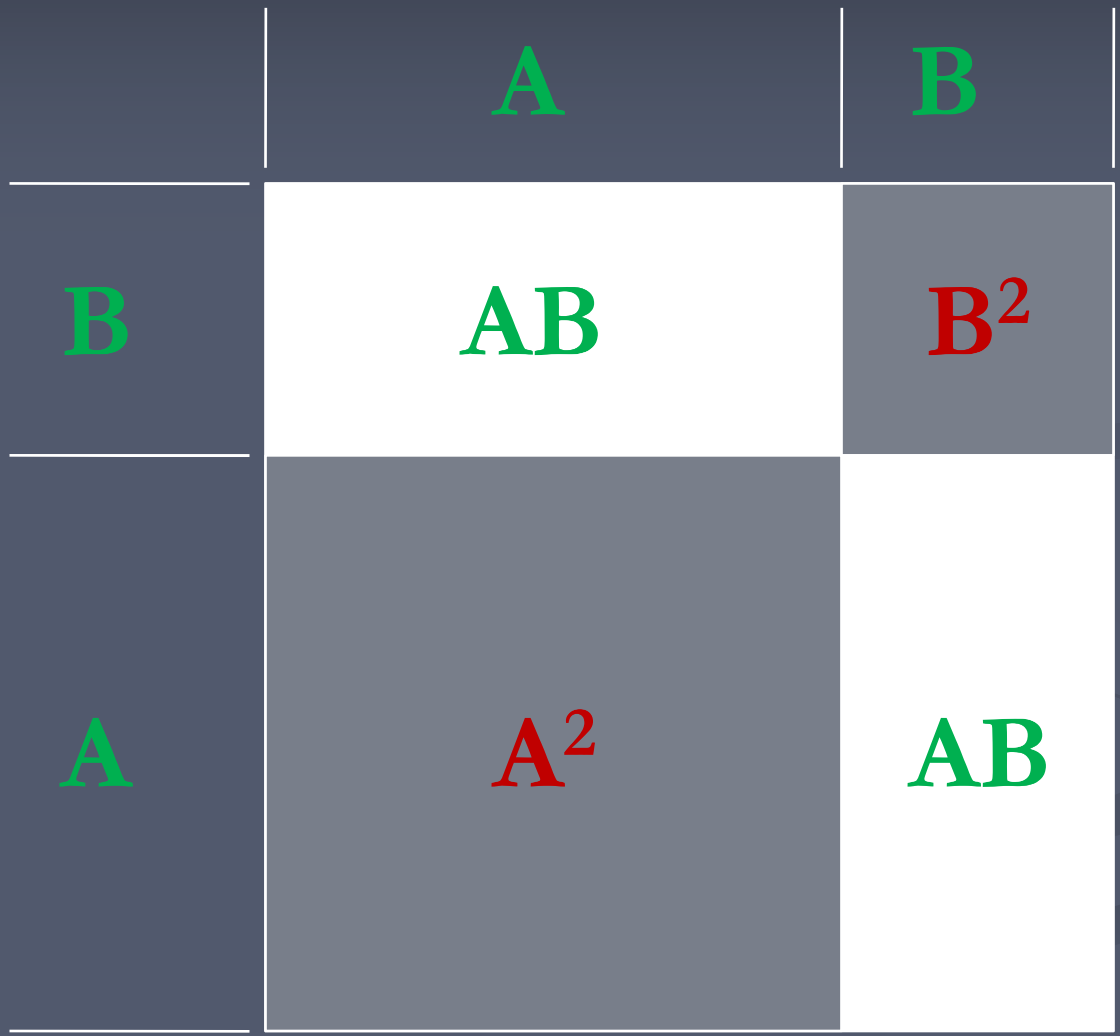
- Babylonia – 1800 BC
  - First known use of algebraic equations
  - Number system was base-60
  - Multiplication table impractical (3600 entries to remember!)
  - Used table of squares to multiply any two integers

  - And this equation:

$$ab = \frac{(a+b)^2 - a^2 - b^2}{2}$$

History

- Babylonia – 1800 BC



History

- Greece – 300 BC
  - Euclid wrote Elements
  - Covered Geometry
    - From "Geo" meaning "Earth"
    - And "Metric" meaning "Measurement"
  - Geometry was the study of Earth Measurements
  - Extended: Earth to space, space-time and beyond
- Alexandria – 50 AD
  - Heron tried to find volume of a frustum, but it required using the square root of a negative number

History

- Persia – 820
  - Al-Khwarizmi wrote a mathematics text based on…
  - Al-jabr w'al-Muqabala
  - Al-jabr: "Reunion of broken parts"
  - W'al-Muqabala: "through balance and opposition"
- Pisa – 1202
  - Leonardo Fibonacci introduces the method to Europe
  - Name is shortened to Al-jabr
  - Westernized to Algebra

History

- 1637 – René Descartes
  - Coined the term "imaginary number"
- 1777 – Leonard Euler
  - Introduced the symbol *i* for imaginary numbers
- 1799 – Caspar Wessel
  - Described complex numbers geometrically
  - Made them acceptable to mainstream mathematicians
- 1831 – Carl Gauss
  - Discovered that complex numbers could be written a + *i* b

History

- 1843 – Rowan Hamilton
  - Discovered quaternions (3D complex numbers)
  - Coined the term "vector" to represent the non-scalar part
  - Invented dot and cross products
- 1844 – Hermann Grassmann
  - Exterior product (generalization of cross product)
- 1870 – William Kingdon Clifford
  - Generalized complex numbers, dot and cross products
  - Died young, and his approach didn't catch on
  - Variants of his approach are often called Clifford Algebras
- 1966 – David Orlin Hestenes
  - Rediscovered, refined and renamed "Geometric Algebra"
  - Claims that "Geometric Algebra" is the name Clifford wanted

Geometric Algebra Introduction

How do we represent direction?

Direction is similar to a unit, like mass or energy, but even more similar to - and +

Pick a symbol that represents one unit of direction – like i.

We live in a 3D universe, so we need three directions: i, j and k

Geometric Algebra Introduction

How do we add direction?

i+i = 2i **Parallel directions combine**

i+j = i+j **Orthogonal directions do not combine**

No matter how many terms we add together, we'll wind up with some combination of i, j and k:

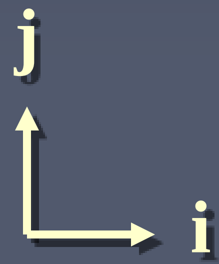
(a**i** + b**j** + c**k**)

We call these vectors…

Geometric Algebra
Simplifying Products

How do we multiply orthogonal directions?

$(\mathbf{i})(\mathbf{j}) = \mathbf{ij}$



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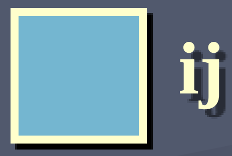
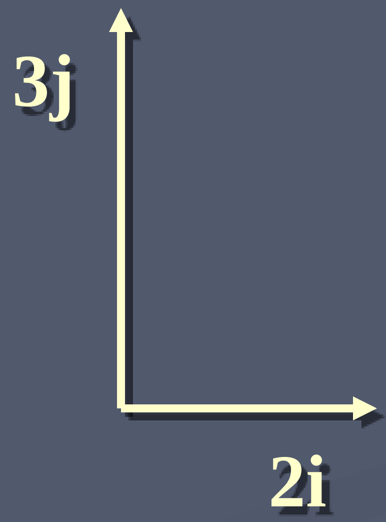


$(2\mathbf{i})(3\mathbf{j}) = 6\mathbf{ij}$



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We call these bivectors…

Geometric Algebra Introduction

What are the unit directions in 2D & 3D?

	2D		3D			
Scalar:	1			1		
Vector:	i	j	i	j	k	
Bivector:	ij		jk	ki	ij	
Trivector:				ijk		

Note: In 3D, there are three vector directions and three bivector directions – this leads to confusion!

Geometric Algebra Introduction

Which are vectors? Which are bivectors?

Velocity
Angular velocity
Force
Torque
Normal
Direction of rotation
Direction of reflection
Cross product of two vectors
The vector portion of a quaternion

Geometric Algebra Introduction

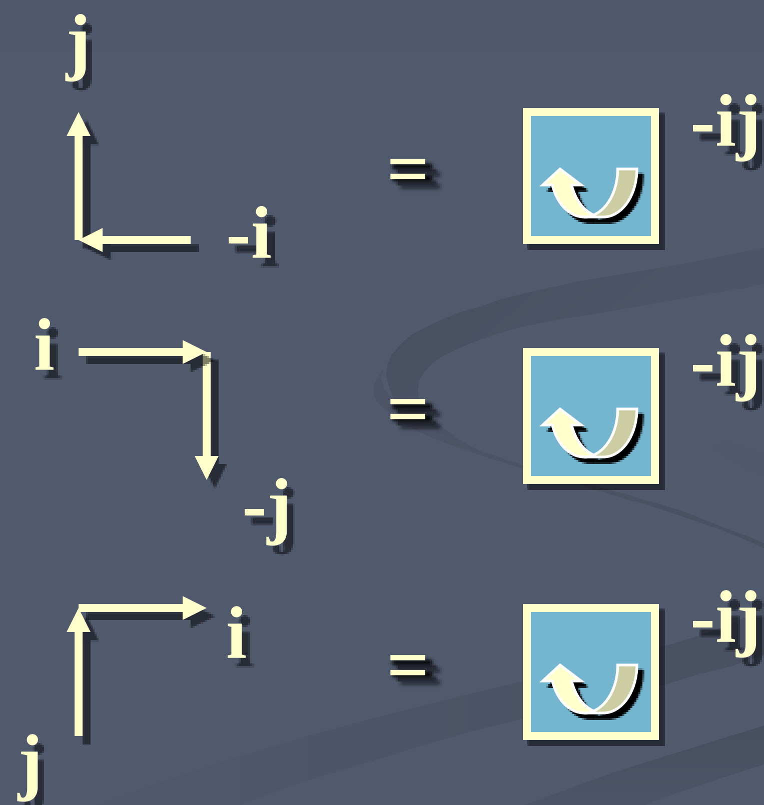
What does a negative bivector represent?

$$(-\mathbf{i})(\mathbf{j}) = -\mathbf{ij}$$

$$(\mathbf{i})(-\mathbf{j}) = -\mathbf{ij}$$

$$(\mathbf{j})(\mathbf{i}) = -\mathbf{ij}$$

$$\mathbf{ji} = -\mathbf{ij}$$



Geometric Algebra Introduction

How do we multiply parallel directions?

ii=?

i represents direction, like 1 or -1 on a number line

(1)(1) = 1

(-1)(-1) = 1

(i)(i) = 1

ii = 1 **i is its own inverse!**

Geometric Algebra
Introduction

How do we multiply directions?

Rule 1: If i and j are orthogonal unit vectors, then:

$$ji = -ij$$

Rule 2: For any unit vector i:

$$ii = 1$$

Just like the **cross** product!

That's it! Now we can multiply arbitrary vectors!

Geometric Algebra
Simplifying Products

Try simplifying these expressions…

iijj	**1**
ijik	**-jk**
ijkjkij	**-j**
kjijk	**i**

Geometric Product

How do we multiply two vectors?

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\ (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

Geometric Product

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \begin{array}{l} + ax\mathbf{ii} + ay\mathbf{ij} + az\mathbf{ik} \\ + bx\mathbf{ji} + by\mathbf{jj} + bz\mathbf{jk} \\ + cx\mathbf{ki} + cy\mathbf{kj} + cz\mathbf{kk} \end{array}$$

Geometric Product – Simplify

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \begin{array}{l} + ax\mathbf{ii} + ay\mathbf{ij} + az\mathbf{ik} \\ + bx\mathbf{ji} + by\mathbf{jj} + bz\mathbf{jk} \\ + cx\mathbf{ki} + cy\mathbf{kj} + cz\mathbf{kk} \end{array}$$

$\mathbf{ii} = 1$

$\mathbf{jj} = 1$

$\mathbf{kk} = 1$

Geometric Product – Simplify

$\mathbf{ii} = 1$

$\mathbf{jj} = 1$

$\mathbf{kk} = 1$

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \begin{array}{l} + ax + ay\mathbf{ij} + az\mathbf{ik} \\ + bx\mathbf{ji} + by + bz\mathbf{jk} \\ + cx\mathbf{ki} + cy\mathbf{kj} + cz \end{array}$$

Geometric Product – Simplify

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$
\begin{aligned}
= \quad & + ax + ay\mathbf{ij} + az\mathbf{ik} \\
& + bx\mathbf{ji} + by + bz\mathbf{jk} \\
& + cx\mathbf{ki} + cy\mathbf{kj} + cz
\end{aligned}
$$

$\mathbf{ji} = -\mathbf{ij}$

$\mathbf{ik} = -\mathbf{ki}$

$\mathbf{kj} = -\mathbf{jk}$

Geometric Product – Simplify

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \begin{aligned} &+ ax + ay\mathbf{ij} + az\mathbf{ik} \\ &- bx\mathbf{ij} + by + bz\mathbf{jk} \\ &+ cx\mathbf{ki} + cy\mathbf{kj} + cz \end{aligned}$$

$\mathbf{ji} = -\mathbf{ij}$

$\mathbf{ik} = -\mathbf{ki}$

$\mathbf{kj} = -\mathbf{jk}$

Geometric Product – Simplify

$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$$= \begin{array}{l} + ax + ay\mathbf{ij} - az\mathbf{ki} \\ - bx\mathbf{ij} + by + bz\mathbf{jk} \\ + cx\mathbf{ki} + cy\mathbf{kj} + cz \end{array}$$

$\mathbf{ji} = -\mathbf{ij}$

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Geometric Product – Simplify

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \begin{array}{l} + ax + ay\mathbf{ij} - az\mathbf{ki} \\ - bx\mathbf{ij} + by + bz\mathbf{jk} \\ + cx\mathbf{ki} - cy\mathbf{jk} + cz \end{array}$$

$\mathbf{ji} = -\mathbf{ij}$

$\mathbf{ik} = -\mathbf{ki}$

$\mathbf{kj} = -\mathbf{jk}$

Geometric Product – Group Terms

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \begin{array}{l} + ax + ay\mathbf{ij} - az\mathbf{ki} \\ - bx\mathbf{ij} + by + bz\mathbf{jk} \\ + cx\mathbf{ki} - cy\mathbf{jk} + cz \end{array}$$

$$= (ax + by + cz) + \ldots$$

Geometric Product – Group Terms

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \begin{array}{l} + ax + ay\mathbf{ij} + az\mathbf{ik} \\ - bx\mathbf{ij} + by + bz\mathbf{jk} \\ - cx\mathbf{ik} - cy\mathbf{jk} + cz \end{array}$$

$$= (ax + by + cz) + (bz - cy)\,\mathbf{jk} + (cx - az)\,\mathbf{ki} + (ay - bx)\,\mathbf{ij}$$

Geometric Product
Inner & Outer Products

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \begin{array}{l} + ax + ay\mathbf{ij} + az\mathbf{ik} \\ - bx\mathbf{ij} + by + bz\mathbf{jk} \\ - cx\mathbf{ik} - cy\mathbf{jk} + cz \end{array}$$

$$= (ax + by + cz) + (bz - cy)\,\mathbf{jk} + (cx - az)\,\mathbf{ki} + (ay - bx)\,\mathbf{ij}$$

$$= \mathbf{A} \cdot \mathbf{B} + \ldots$$

Geometric Product
Inner & Outer Products

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \begin{aligned} &+ ax + ay\mathbf{ij} + az\mathbf{ik} \\ &- bx\mathbf{ij} + by + bz\mathbf{jk} \\ &- cx\mathbf{ik} - cy\mathbf{jk} + cz \end{aligned}$$

$$= (ax + by + cz) + (bz - cy)\,\mathbf{jk} + (cx - az)\,\mathbf{ki} + (ay - bx)\,\mathbf{ij}$$

$$= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B}$$

Geometric Product
Inner & Outer Products

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \begin{array}{l} + ax + ay\mathbf{ij} + az\mathbf{ik} \\ - bx\mathbf{ij} + by + bz\mathbf{jk} \\ - cx\mathbf{ik} - cy\mathbf{jk} + cz \end{array}$$

$$= (ax + by + cz) + (bz - cy)\,\mathbf{jk} + (cx - az)\,\mathbf{ki} + (ay - bx)\,\mathbf{ij}$$

$$= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B}$$

Geometric Product Inverse

Vectors can be inverted!

$$aa^{-1} = 1$$

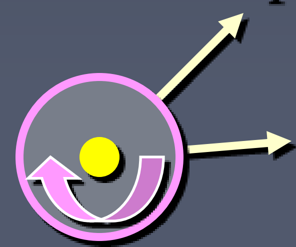
$$aaa^{-1} = a$$

$$(aa)a^{-1} = a$$

$$a^{-1} = \frac{a}{aa}$$

2D Product

$\mathbf{A} = a\mathbf{i} + b\mathbf{j}$



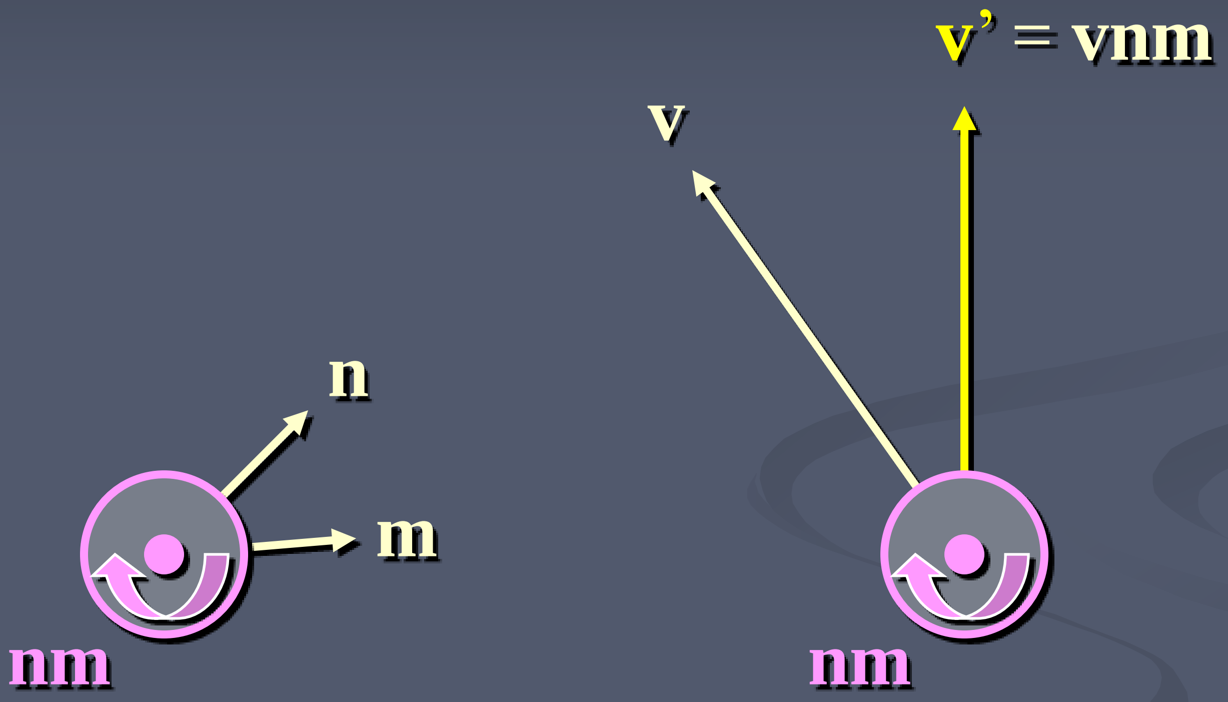
$\mathbf{B} = x\mathbf{i} + y\mathbf{j}$

$$\mathbf{AB} = (ax + by) + (ay - bx)\,\mathbf{ij}$$

Point-like "vector" (a scalar)
Captures the parallel relationship between **A** and **B**

Plane-like "vector" (a bivector)
The perpendicular relationship between **A** and **B**

2D Rotation



$v' = v(nm) = vnm$

Note: This only works when **n**, **m** and **v** are in the same plane!

A Complex Connection

What is the square of **ij**?

$$(\mathbf{ij})(\mathbf{ij}) = (-\mathbf{ji})(\mathbf{ij}) = -\mathbf{jiij} = -\mathbf{j}(\mathbf{ii})\mathbf{j} = -\mathbf{jj} = -1$$

$$(\mathbf{ij})^2 = -1 \quad !!!$$

$$\mathbf{ij} = \sqrt{-1} \quad !!!$$

$$\mathbf{ij} = i \quad !!!$$

So… $(a + b\mathbf{ij})$ is a complex number **!!!**

The Complex Connection

This gives a geometric interpretation to imaginary numbers:

Imaginary numbers are bivectors (plane-like vectors).

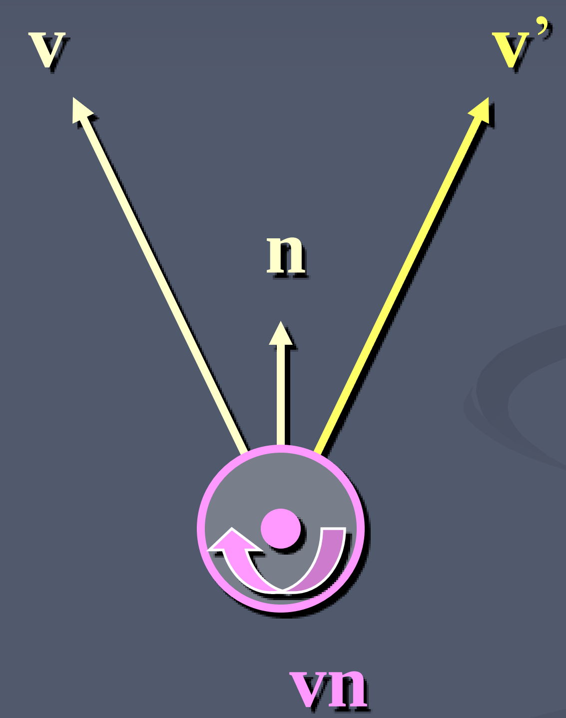
The planar product (“2D cross product”) of two vectors.

The full geometric product of a pair of 2D vectors has a scalar (real) part “•” and a bivector (imaginary) part “”.

It’s a complex number! (2 + 3**ij** is the same as 2 + 3*i*)

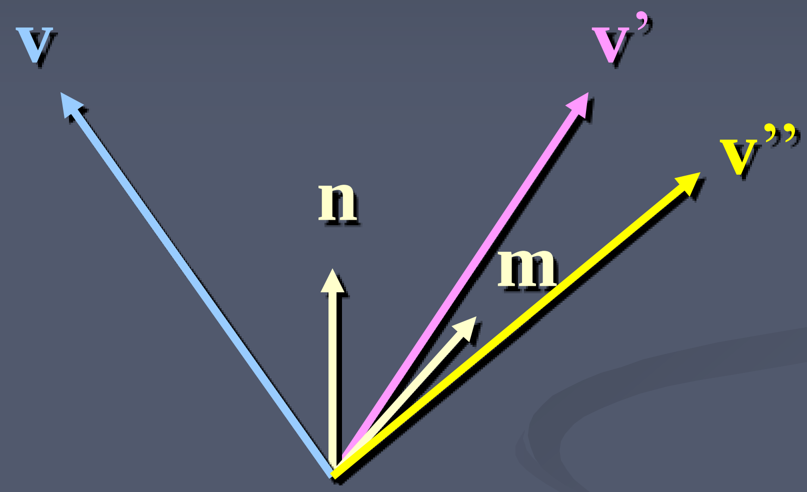
So, complex numbers can be used to represent a 2D rotation. This is the major reason they appear in real-world physics!

2D Reflection



$v' = n(vn) = nvn$

Rotation by Double Reflection



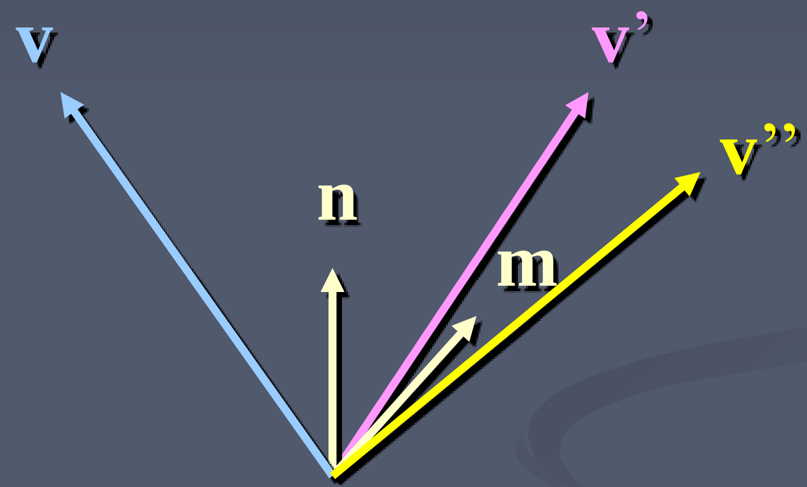
$$\mathbf{v}' = \mathbf{n}\mathbf{v}\mathbf{n}$$

$$\mathbf{v}'' = \mathbf{m}(\mathbf{v}')\mathbf{m} = \mathbf{m}(\mathbf{n}\mathbf{v}\mathbf{n})\mathbf{m} = \mathbf{m}\mathbf{n}\mathbf{v}\mathbf{n}\mathbf{m} = (\mathbf{m}\mathbf{n})\mathbf{v}(\mathbf{n}\mathbf{m})$$

Angle of rotation is equal to twice the angle between **n** and **m**.

Note: Rotation by double reflection works in any dimension!

Rotation by Double Reflection



$$v' = nvn$$

$$v'' = m(v')m = m(nvn)m = mnvnm = (mn)v(nm)$$

Angle of rotation is equal to twice the angle between **n** and **m**.

The product mn is a quaternion and nm is its conjugate!